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Research Article

The role of Difference Sequence in learning the concept of the derivative using the ACE cycle

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This study examines learning of derivative concept by using the ACE (Activities, Class discussions, and Exercises) teaching cycle. In this study, the concept of discrete derivative approach was used, in which a step-by-step method of differential sequence for functions defined on Z and Q is proposed. The purpose of this study was the effectiveness of using ACE teaching cycle in learning the concept of derivative. The present study was conducted using the pre- and post-test research design. This study was conducted on 42 university students, 21 people from each of two experimental and control groups were chosen using real sampling and were distributed at random. The concept of the discrete derivative approach was taught to the experimental group using the ACE cycle in the Geogebra software environment and to the control group in a routine manner. A pre-test was taken at the beginning of the training course, and a post-test was taken from both groups at the end. The data were analyzed with an independent two-group *t*-test. The findings showed that the teaching approach with the use of the ACE teaching cycle in the Geogebra software environment facilitate on students' comprehension of the derivative concept. Therefore, the ACE teaching cycle, with the help of the software, can help in teaching calculus, especially the concept of derivative, to develop students' conceptual understanding.

Keywords: ACE teaching cycle, derivative, discrete approach, teaching and learning

1. Introduction

Calculus is one of man's most outstanding achievements (NCTM, 2000) which has played an important role in human civilization. The derivative is one of the essential basic concepts in differential and integral calculus, which is related to function, limit, and rate of change and has many applications in various sciences. In other words, the derivative is the heart of modern mathematics. This concept is given special attention because it is used in learning and teaching concepts such as antiderivative and integral. Forming the derivative concept, the learning process, and its progress can help solve students' problems. Accordingly, it is possible to gain knowledge that will help teachers and professors to better teach this concept and other mathematical concepts in the direction of better and more stable learning for learners. There are three different interpretations of the concept of derivative. The first interpretation of the derivative is the slope of the tangent line to the curve at point $A = (a, f(a))$. Another interpretation of the derivative is the instantaneous rate of change of the function. In contrast, the physical meaning of the derivative refers to the speed and acceleration of a moving object at a moment in time. The third interpretation of the derivative as a function is where the formal definition of the function's derivative $f(x)$ at the point $x = a$ is equal to $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. The derivative is difficult for most students to understand the formal meanings of the limit, so its not possible accurately apply the definitions in different situations, can only solve intuitive problems, and do not have a deeper understanding of the concepts. Research has shown that Derivation is one of the most difficult concepts for students due to its complex definition and expression (Thompson, 1994; Zandieh, 2000). Some students do not get a proper conceptual understanding of the derivative even after completing calculus courses (Eisenherg, 1992). Most students misunderstand the slope of the line and the tangent to the curve and their relationship (Asiala et al., 1997; Feudel & Biehler, 2021; Feudel, 2019; Ubuz, 2007). Most learners can coordinate and communicate between the two features of the derivative function and its primary function (Haciomeroglu & Chicken, 2012). Previous research on derivation concepts has shown that most students have little conceptual or intuitive knowledge of derivations, even though they have sufficient procedural knowledge of derivations increase (Asiala et al., 1997; Clark et al., 1997; Dominguez et al., 2017; Orton, 1983; Tall, 1993; Thompson, 1994; Zandieh, 2000). According to the research mentioned the difficulties of understanding the concept of derivative, Weigand (2014) introduces a new approach to understanding the concept of derivative. Weigand (2014) we present a stepwise discrete approach to better understand the concept of derivation by continuous differentiation of functions defined in the discrete domains Z and Q and subsequent generalization to the R domain. The advantage of this method is that it does not present the concept of limit at the beginning of the work, and the concept of the rate of change is expressed using discrete examples. Therefore, discrete Zn functions are presented to interpret the function f's derivative at a specific point of the graph f and, as a result, calculate the instantaneous level of modification in the discrete approach of the sequence of decisive gradients.

APOS (Actions, Processes, Objects, and Schemas) theory explores students' understanding of mathematical concepts. According to this theory, to understand mathematical concepts, learners must have a mental structure about actions, processes, objects, and schemes. Without such a mental structure, learning this concept is almost impossible (Arnon et al., 2014). APOS Theory's educational strategy for teaching mathematical concepts is called the ACE Cycle and includes computer-based activities, classroom discussions, and exercises. The purpose of this study was to answer the following questions:

What is the effect of using the discrete approach by using the ACE teaching cycle in the Geogebra software environment on the improved comprehension of students to learn the concept of the derivative?

2. Literature Review

Research using the APOS-ACE theory has shown that the computer-aided ACE cycle can help students' understanding of mathematical concepts. Borji and Martinez-Planell (2019) used the APOS-ACE theory to investigate students' comprehension of the implicit function and its derivative. First, a genetic decomposition (GD) was proposed to understand the implicit function and its derivative, and then it was designed and implemented using the proposed genetic decomposition of the ACE cycle to help students make mental constructs they lacked. The results showed that the ACE teaching cycle was effective in promoting students' understanding of implicit functions and their derivation.

The APOS-ACE theory was employed by Borji et al. (2018) in an effort to enhance students' understanding of function derivatives visually. The ACE cycle, which was created using the Maple software, was taught to the experimental group for this purpose, whereas the control group learned about derivatives the old-fashioned way, through lectures. By contrasting the performance of the experimental group with that of the control group, the effectiveness of this training was evaluated. The findings showed that the experimental group students better understood the concept of derivatives than the control group.

Borji and Voskoglou (2017) designed an ACE cycle for instructing polar coordinates. The designed ACE cycle was employed on the students of the experimental group in one of Iran's universities. Results from the experimental group and the control group were contrasted, in which the control group was taught polar coordinates in a traditional lecture-based way to investigate the efficiency of the ACE cycle. The outcomes showed that the experimental group students had a better and more appropriate understanding of polar and Cartesian coordinates than the control group. Siyepu (2013) used APOS theory to analyze students' mistakes in learning exponential, logarithmic, and trigonometric derivatives. Using a case study approach, we investigated twenty students participated in a graduate school program at a technical university. As a result, we found that most students' level of understanding is in the action phase, or the contradiction between the action phase and the process of APOS theory. The ACE teaching cycle should be designed and implemented to enable students to develop the schemes they need.

Cetin (2009) used the APOS theory to examine first-year university students' understanding of the concept of limit. The teaching method was implemented based on teaching experiences and with the help of APOS theory ideas and a case study to answer the research questions. A total of 25 first-year students of one of the universities in Turkey who had the general mathematics course 1 in the fall semester of 2007-2008 participated in this research. Every week, the students worked in groups for 3 hours in the computer lab and attended the class for 6 hours. In the computer lab, programming activities related to the concept of limit were worked on before the concept of limit was taught to them in the primary classroom. Open-ended questions were designed as a pre-test and post-test and were taken from students before and after teaching to check the changes in their understanding. At the end of teaching, interactive sessions were done with students to gather qualitative info. The data were quantitatively and qualitatively analyzed by the researcher under the theory of APOS. The results showed that the students' thinking was consistent with what was predicted in the genetic analysis. In addition, the teaching given to the students played an important role in developing their comprehension of the concept of limit. Weller et al. (2009) designed a decimal repetition unit for teachers to understand the relationship between rational numbers (fraction or integer) and their decimal expansion using the APOS theory pedagogy strategy in the ACE teaching cycle (activities, class discussions, and exercises). Compared to the control group, the ACE teaching cycle group has made significant progress understanding the relationship between rational numbers and their decimal expansions. Asiala et al. (1997) used the APOS theory to analyze students' graphic apprehension of the derivative concept. Interactive sessions on derivatives were conducted c. Students trained through this ACE teaching cycle demonstrated strong process understanding in understanding the symbol f(x) and interpreting the relationship between the derivative, its graph, and the function's graph.

3. Background

This section describes the two theoretical frameworks used in the research (discrete derivative approach, APOS theory).

3.1. Discrete Derivative Approach

Weigand (2014) introduced the concept of a discrete approach to calculus where, by considering discrete functions, an average rate of change is proposed based on various outcomes. Weigand (2014) stated that working with the discrete sequence and its differential sequence can be a basic idea for developing an understanding of the concepts of variational and derivative functions. Weigand (2014) stated about the discrete approach that "In developing this first level of conception, the concept of boundaries is only used in an intuitive sense. All computations can be performed at the discrete algebraic level. This discrete approach to the notion of derivatives is a preparation for understanding the derivatives of real functions. This concept provides an improved comprehension of the meaning of a variation or differential fraction, implements limits or approximation processes by clearly Manipulating sequences (or discrete functions) in this sense is beyond intuitive understanding of the limits". Weigand (2014) introduces five levels for applying the discrete approach to understand the concept of derivative.

3.1.1. First level: differential sequences

The purpose of this level is to familiarize the concept of sequential modification of $\Delta a_n = a_{n+1} + a_n (n \in N)$. If $\Delta n = 1$ can be considered as a rate of change used in real-life problems, such as the average air temperature each year, which may be shown in a table or graph.

3.2.2. Second level: the concept of (Quadratic) Z-functions

Previously, the sequence was defined with domain N. Now the notion of the sequence is extended to functions defined on Z. The function f: Z→R is called the Z function. The functions y = f(z) are extended sequences defined on the numbers z∈ Z.

Example 1: For the function $f(z) = z^2 - 2z + 3$, the differential Z function is as follows: $D_f(z) = f(z+1) - f(z) = 2z + 1 - 2$

Example 2: For the function $f(z) = az^2 + bz + c, D_f(z) = 2z + a + b$. This can be obtained to find that D_f(z) is not dependent on the c parameter.

3.2.3. Third level: polynomial Z-function

This concept can be extended to polynomial functions of a higher degree.

Example 3: For the function $f(z) = az^3 + bz^2 + cz + d$ we have $D_f(z) = 3az^2 + (3a + 2b)z + a + b + c$. D_f is a quadratic function that does not depend on the parameter d.

2.3.4. Fourth level: Exponential function

Example 4: For the exponential function $E(z) = a^z (a \in R^+, z \in Z)$. The difference function is $D_E(z) = E(z+1) - E(z) = a^{z+1} - a^z = a^z(a-1) = E(z)(a-1)$. To obtain D_E(z), it is sufficient to multiply $E(z)$ by the factor $(a - 1)$.

2.3.5. Fifth Level : Transferring ideas from Z to Q and R

This section selects the following domains, a subset of [-2,2].

$$z_k = \{\dots, -\frac{2}{k}, -\frac{1}{k}, 0, \frac{1}{k}, \frac{2}{k}, \dots\}$$

$$z_{10} \in Z_{10} = \left\{\dots, -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, \dots\right\} \left(Z_{10} = \frac{z}{10}, z \in Z\right).$$

$z_{10} \in Z_{10}$ and f function is described as $f_{10}: Z_{10} \rightarrow R$. The intervals are limited to $1/10$ instead of 1 units to achieve the rate of change of successive values, and the differential fraction function Z_{10} is obtained:

$$D_{f(n)}(z_n) = \frac{f\left(z_n + \frac{1}{n}\right) - f(z_n)}{\frac{1}{n}} \quad (z_n \in Z_{10}) = \{\dots, -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, \dots\}$$

This problem can be generalized to 1/n in the longitudinal distances $n \in N$, and the function Z_n is the differential fraction of $D_{f(n)}$.

$$D_{f(n)}(z_n) = \frac{f\left(z_n + \frac{1}{n}\right) - f(z_n)}{\frac{1}{n}} \quad (z_n \in Z_n) = \{\dots, -\frac{2}{n}, -\frac{1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \dots\}$$

Example 5: For the quadratic function z, we have $D_f(z_n) = 2az_n + b + \frac{a}{n}$. Now, if $n \rightarrow +\infty$: $D_f(z_n) = 2az + b$ which is the derivative of the function. This idea is similar to the exponential Z-function ($E(z_{10}) = a^{z_{10}}$), and its differential fraction function is:

$$D_E(z_{10}) = \frac{E\left(z_{10} + \frac{1}{10}\right) - E(z_{10})}{\frac{1}{10}} = \frac{a^{z_{10}+\frac{1}{10}} - a^{z_{10}}}{\frac{1}{10}} = a^{z_{10}} \frac{a^{\frac{1}{10}} - 1}{\frac{1}{10}}$$

If $\frac{a^{\frac{1}{10}}-1}{\frac{1}{10}} = 1$ and/or $a = \frac{2}{5937}$, $E(z_{10})$ will equal to $D_E(z_{10})$. Considering the exponential function $E(z_n) = a^{z_n}$:

$$D_E(z_n) = \frac{E\left(z_n + \frac{1}{n}\right) - E(z_n)}{\frac{1}{n}} = \frac{a^{z_n+\frac{1}{n}} - a^{z_n}}{\frac{1}{n}} = a^{z_n} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}}$$

But when $n \to \infty$, $\lim_{n\to\infty} \frac{a^{\frac{1}{n}}-1}{\frac{1}{n}} = 1$. This means that it $D_E(z_n)$ is the derivative of the exponential function. In the following, the instantaneous change song is performed for $x_0 \in D \subseteq \mathrm{R}$. We form the differential fraction sequence for the real difference of the value.

$$D_n(x_0) = \frac{f\left(x_0 + \frac{1}{n}\right) - f(x_0)}{\frac{1}{n}} \quad (n \in N)$$

Example 6: At the point $(x_0, f(x_0))$ for the function $f(x) = ax^2 + bx + c$, we have:

$$D_n(x_0) = \frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{\frac{1}{n}} = \frac{a\left(x_0+\frac{1}{n}\right)^2+b\left(x_0+\frac{1}{n}\right)+c-ax_0^2-bx_0-c}{\frac{1}{n}} = 2ax_0 + b + \frac{a}{n}$$

Therefore, the sequence $D_n(\mathrm{x}_0)$ is interpreted according to the diagram f the decisive night sequence at one point.

3.2. APOS theory

As has been proven, APOS theory helps to elaborate the structure of some mathematical concepts, and established APOS theory helps to elaborate the structure of some mathematical concepts. APOS theory is according to Piaget's principle that a person uses certain mental mechanisms to build mental structures to learn different concepts, including mathematical concepts, and then uses these structures to solve problems in mathematics. According to this principle, a person can create a mental structure for each mathematical concept that is suitable for that concept and can be used for understanding, learning, and applying that concept (Arnon et al., 2014). The following central concept is used in APOS theory:

3.2.1. Concept of action

An action transforms a mathematical object that has already been constructed and is understood as an external component. This is an external operation because each transformation step must be explicitly performed by an external instruction. Furthermore, each stage creates the next, so action steps cannot yet be imagined, and none can be excluded (for more on this, see Arnon et al., 2014).

The most basic understanding of the derivative is its concept as an action, a repetition of mental acceptance or physical manipulation of objects by some researchers. At this stage of learning, students recognize derivative as a law. Therefore, a person at the action level, except for calculating the derivative rule (at specific points) and command with rules, cannot do more. Understanding the concept of the derivative of quadratic functions does not go beyond the scope of a practical understanding of the derivative of functions. All the mentioned cases require process or object perceptions or both together. Learners who can perform the procedures have understood the derivative in the form of action. For example, it can calculate the value f(2) by having the rule $f(x) = x^2 - 3x$. Therefore, learners who have a practical understanding of the derivative of a function need an expression like $'(x)$, and when the calculation of the derivative of the function $f(x) = x^3$ requires clear math like, it will only be able to write $f'(x) = 3x^2$.

3.2.2. Concept of process

When the learner reflects on the repeated action, it becomes Interiorization as a process. At this time, the learner has an internal structure to perform the same activity. An external stimulus does not necessarily guide an internal system that performs the same action, and the control is in the hands of the learner. There is a significant relationship with other mathematical knowledge, which allows process imagination and predict the results without the need for explicit implementation. Essential relationships in a process concept allow linking various demonstrations and process justifications. Other ways can be adjusted to form new processes. A process can be created by internalizing various activities and coordinating the subsequent process. Compared to action, a process is understood as something internal and under the learner's control rather than a reaction to external stimuli. Generally, it is a practical process that happens entirely in the mind.

The next level of understanding the concept of derivative is process. This concept is a deeper comprehension of the concept of derivative, as something that takes something transforms it, produces something entirely new, gives it, and involves an explicit formula or rule. At this level of learning, students are willing to accept functions that involve fuzzy transformations.

3.2.3. Concept of the object

Suppose the learner can reflect on the actions that have been applied to a specific process and find out that each process can be considered a whole that can be changed (actions or processes). If these changes are made, it is said that the learner has understood the process in the form of an object. At this stage, the process is summarized as an object. A process or action on an object often requires splitting the object into the process from which it was derived.

Summarizing processes into objects and splitting objects into their constituent processes can be seen in manipulations with derivatives of fixed and radical functions. For example, suppose a learner can write the derivative of the sum of several functions as the sum of the derivative of several functions. In that case, he has understood the function's derivative as an object. Understanding the formal definition of mathematical concepts, such as function, limit, and derivative, is at this level.

3.2.4. Scheme

A cohesive combination of behaviors, procedures, objects, and other schemas connected to a particular mathematical notion is known as a mathematical schema. The schema is consistent in that the various components are connected in such a way that you can determine which issues are relevant to the schema.

At this level, the learner can determine the critical and extremum points of the function and understand when $f'(x) = 0$. The derivative schema may include another schema, such as the function schema or the limit schema. The derivative scheme may also include different methods of the derivation of functions. In this case, one of the actions that can be applied to this schema is choosing the appropriate derivation method for the given problem.

Genetic Decomposition [GD] Concepts related to APOS theory for analyzing students' understanding of mathematical concepts. Based on the Arnon et al. (2014), genetic analysis is a hypothetical model that explains the mental structures and mechanisms necessary for students to learn specific mathematical concepts. Genetic decomposition is described in terms of this theory's mental structures (action, process, object, and schema) and mechanisms (internalization, compression, and coordination). The genetic breakdown is not unique and may be presented as a concept.

3.3. ACE cycle

ACE teaching cycle is a pedagogical strategy commonly used in APOS theory based on cooperative learning Activities (A) and genetic decomposition, usually comprising students' usage of computers. Class discussions (C) help reflect and institutionalize the mathematics learned and Exercises (E) (Arnon et al., 2014).

3.3.1. Activities

The educational process of the ACE cycle takes place in the first step of the classroom in the computer workshop, where students participate exclusively as a team in completing tasks based on a computer program that has been prepared in advance to cultivate their mental constructions. Compared to other activities, the critical point in these activities is to pay attention to the heuristic nature of learning. According to Arnon et al. (2014) adopting a mathematical programming language can help students understand mathematical concepts more thoroughly. The activities planned in this section for teaching are based on the same genetic analysis for constructing a specific concept for teaching.

3.3.2. Class discussions

Class discussions should be performed in the classroom after the educational implementation in the computer workshop and doing the activities. Students are grouped in teams in the class and reflect on and perform the same activities in the computer workshop. The teacher helps the groups by guiding the plan and providing the opportunity to reflect on the tasks and calculations.

3.3.3. Exercises

Students are assigned regular exercises to work on to internalize the learning of a concept, and they are expected to complete the exercises at home. In order to assist the continued development of the mental constructs identified by genetic analysis, these exercises comprise standardised challenges created to support computer-based activities and classroom discussion (Dubinsky et al., 2013). The objectives of solving the exercises can be summarized as follows:

1) Strengthening the ideas made by students.
2) Reflecting on the concept they have learned.
3) Thinking about the situation they want to study later.

4. Method

4.1. Research Design

This quasi-experimental study was conducted to determine the efficiency of the ACE teaching cycle on students' comprehension of the derivative concept with a discrete approach based on two experimental and control groups, with the experimental group receiving the independent variable "ACE" as well as pre-test and post-test interventions. To compare the outcomes, this study used a pre-test-post-test design with a control group.

4.2. Participants

The study population included all undergraduate computer engineering students in the first semester of one of Iran's universities who were studying in the academic year 2022-2023. 42 participants in total were chosen using a practical sample technique, and the experimental and control groups were randomly split into 2 groups of 21. Every pupil freely took part in the current research.

4.3. Process and Data Collection

First, a pre-test was given to determine the homogeneity of the groups at the start of the semester. The results showed that both groups were homogeneous and equal. Then, a genetic analysis was created for instructing and learning the derivative concept with a discrete approach (Table1). Second, all students in the experimental group participated in a GeoGebra software lab class for 6hrs (1.5 hours every week for 4 weeks). Based on the genetic analysis presented in the first step, the ACE cycle, which was designed using this software for the derivative concept with a discrete approach, was taught in the experimental group. The students in the control group only wrote notes on the whiteboard during ordinary class sessions. Third, after completing the training course, the post-test was performed on both groups, and the results were analyzed by analysis of covariance.

4.3.1. Designing ACE teaching cycle for derivative concept

Cycle 1: Computer-based activities. In the first stage, the classroom was held in the computer workshop during a two-hour session, where the students participated in the tasks based on the computer program in groups (groups of 3 people). The purpose of the computer activities stage was for the students to build the mental structures needed to understand the derivative concept with a discrete approach. The instructor helped them in the computer workshop when the groups had problems writing functions in the Geogebra software or when their program had an error.

Table 1

Genetic Decomposition of derivative concept with the discrete approach

The concepts that are considered so that students can understand the concept of the derivative with a discrete approach are: • A concept of sequence and differential sequence. • Functions defined on the domains Z and Q as well as R. • Differential function of Z-function, operations with functions, and coordination of analytical and geometric representation of Z-function and its differential function. • Differential subtraction function, operation with functions, and coordination of analytical and geometric representation of function $f(z_n)$ and $D_{fn}(z_n)$.
The Z-function and the differential function of the differential subtraction function should be coordinated by obtaining the differential function from Z-functions (quadratic), polynomial Z-functions, and exponential functions. The differential function with domain Z may be considered as the slope of the function between the points $(z, f(z))$ and $(z+1, f(z+1))$. These actions become a process that transfers this idea from the Z-function to the functions defined in the Q subset to create a process that can be thematized in the form of $D_{fn}(z_n) = \frac{f\left(z_n+\frac{1}{n}\right)-f(z_n)}{\frac{1}{n}}, z_n \in z_n = \left\{\dots, -\frac{1}{n}, -\frac{2}{n}, 0, \frac{1}{n}, \frac{2}{n}, \dots\right\}$ differential fraction function with the coordination of analytical and geometrical representations.
We select a constant value of $z_0 \in \mathrm{z_n}$ with a total value of $x_0 \in DCR$ and consider the differential fraction sequence for the real function f according to the value x_0 for n ={1,2,3,.....}. $n \to D_n(x_0) = \frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{\frac{1}{n}}$ and the transformation process required to communicate its various representations is internalized. When this process is extended to the real function f of the fractional differential function $D_n(x_0)$, the concept of the instantaneous rate of change is constructed as an object because of a particular relation between the function $f(x_0)$ and $D_n(x_0)$ is compressed. Students can use a discrete approach to answer any derivative problem when the aforementioned acts, processes, and objects are arranged into a logical schema.

Whenever the program for each activity was written, the students ran their program on several functions to understand the primary purpose of each activity. The first activity focused on obtaining the differential function and drawing the graph of the differential function of Z-functions (quadratic), polynomial Z-functions, and exponential functions with the Z domain. The differential function $Df(z)$ with the domain of Z can be considered as the slope of the function between the points $(z+1, f(z+1)), (z, f(z))$. The details of the first activity are given below.

Task 1: First, the students were instructed to write a program A_1 in Geogebra that receives functions Z (quadratic), polynomial Z, and an exponential with domain Z as input. The outputs were the differential function of the above functions and drawing graphs of the function (z) and their differential function. The aforementioned method is written in GeoGebra by the uses of these commands:

For Z-function (quadratic):

```
A1:Creat slider:a,b,c;(Interval,Min:-8,Max:30,Increment:0/1)
Input:f(z)=az^2+bz+c
Input: Df(z)=f(z+1)-f(z)
Input:sequenc(-8,8)
Input:(L_1,f(L_1))
Input:( L_1,D_f(L_1))
```

For the polynomial Z-function (cubic):

```
A1:Creat slider:a,b,c,d;(Interval,Min:-8,Max:30,Increment:0/1)
Input:f(z)=az^3+bz^2+cz+d
Input: Df(z)=f(z+1)-f(z)
Input:sequenc(-8,8)
```

Input:(L_1,f(L_1))

Input:(L_1,D_f(L_1))

For the exponential function:

A_1:Creat slider:a;(Interval,Min:1,Max:30,Increment:0/1)

Input:E(z)=a^z

Input:D_E(z)=E(z+1)-E(z)

Input:sequenc(-10,10)

Input:(L_1,E(L_1))

Input:(L_1,D_E(L_1))

Task 2: Students were asked to use Program A_1 for the following inputs:

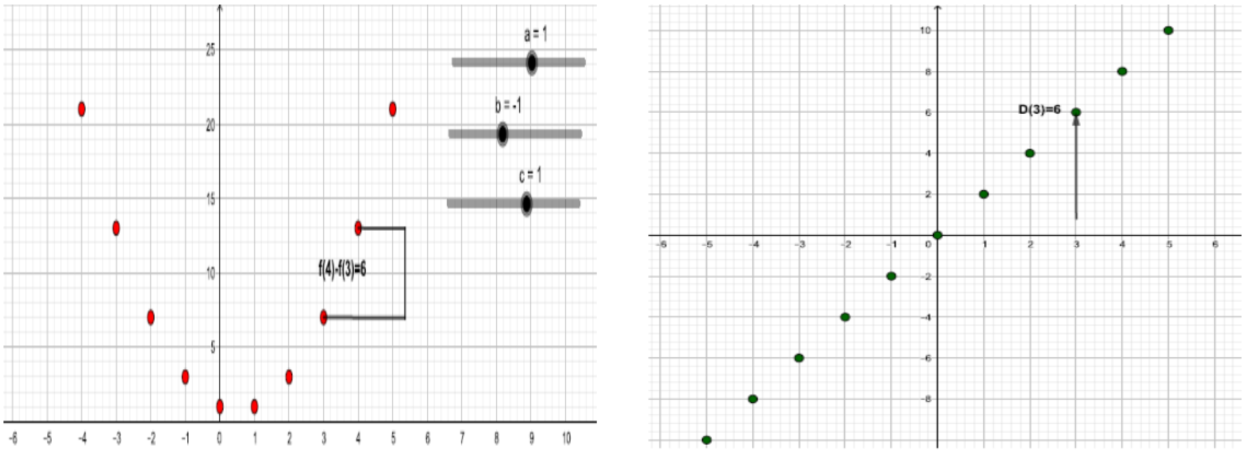
1) $f(z) = z^2 - z + 1$

2) $f(z) = 0/2z^3 - 0/25z^2 - 0/5z + 1$

3) $E(z) = 1.5^z$

Figure 1

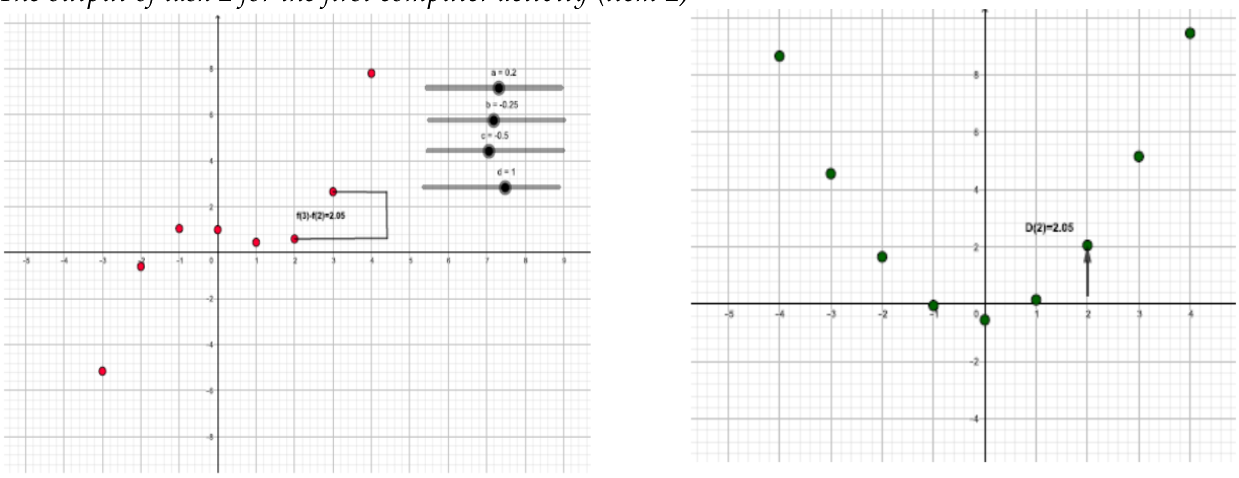
The output of task 2 for the first computer activity (item 1)



$f(z) = z^2 - z + 1$ $\quad$ $Df(z)$=f(z+1)-f(z)

Figure 2

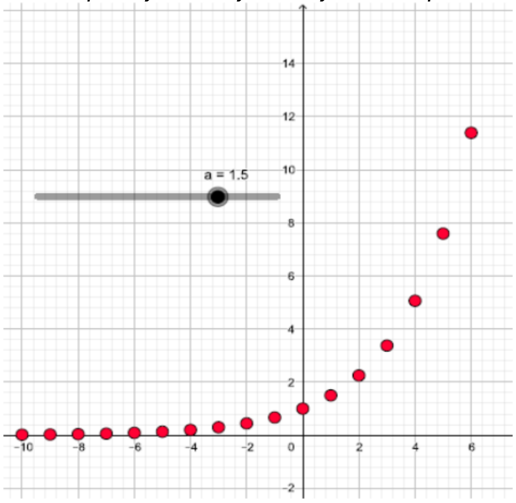
The output of task 2 for the first computer activity (item 2)

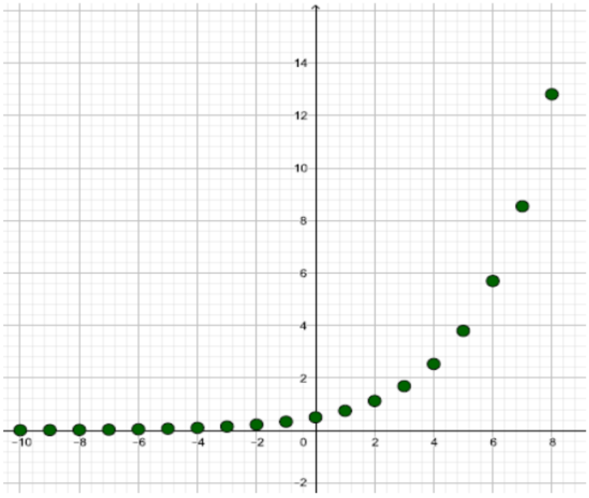


$f(z) = 0/2z^3 - 0/25z^2 - 0/5z + 1$ $\quad$ $D_f(z) = 0/6z^2 + 0/1z - 0/55$

Figure 3

The output of task 2 for the first computer activity (item 3)





$E(z) = 1.5^z$ $D_E(z) = E(z+1) - E(z)$

In the 2nd task, students created a Geogebra approach to graph the differential fraction function of Z-functions (quadratic), polynomial Z-functions, and exponential functions with a domain $z_{10} = \frac{z}{10}(z\epsilon Z)$. At this stage, the idea of the function Z was transferred to the functions defined in the subset Q that form the differential fraction function. The details of the third activity are as follows.

Task 1: The students were instructed to write an A_2 program in Geogebra that receives functions Z (quadratic), polynomial Z, and an exponential with the domain $Z_{10} = \frac{Z}{10}$ as input, and the outputs are the differential fraction function of each of the above functions and draw the graph of the function $f(z_{10})$ and $D_{f10}(z_{10})$. The above method is written in Geogebra as follows:

For Z-function (quadratic):

A_2:Creat slider:a,b,c;(Interval,Min:-8,Max:30,Increment:0/1)

Input:f(z)=az^2+bz+c

Input: $D_f(z)$=f(z+1)-f(z)

Input:sequenc(-8,8,0/1)

Input:(L_1,f(L_1))

Input:(L_1,D_f(L_1))

For the polynomial Z-function (cubic):

A_2:Creat slider:a,b,c,d;(Interval,Min:-8,Max:30,Increment:0/1)

Input:f(z)=az^3+bz^2+cz+d

Input: $D_f(z)$=f(z+1)-f(z)

Input:sequenc(-5,5,0/1)

Input:(L_1,f(L_1))

Input:(L_1,D_f(L_1))

For the exponential function:

A_2:Creat slider:a;(Interval,Min:1,Max:30,Increment:0/1)

Input:E(z)=a^z

Input:D_E(z)=E(z+1)-E(z)

Input:sequenc(-10,10,0/1)

Input:(L_1,E(L_1))

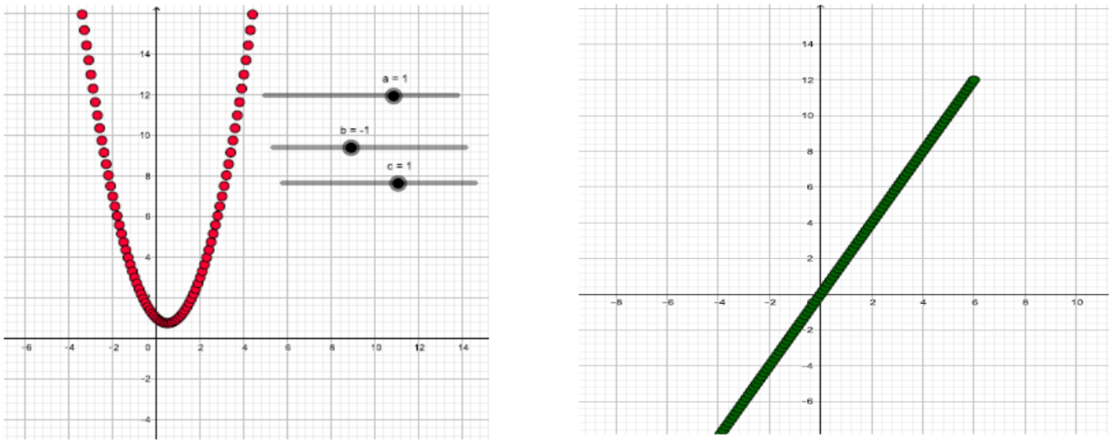
Input:(L_1,D_E(L_1))

Task 2: Students were introduced to employee Program A_2 for the following entries:

1) $f(z_{10}) = {z_{10}}^2 - z_{10} + 1$
2) $f(z_{10}) = {z_{10}}^3 + 3z_{10} + 2$
3) $E(z_{10}) = 1.5^{z_{10}}$

Figure 4

The output of task 2 for the second computer activity (item 1)

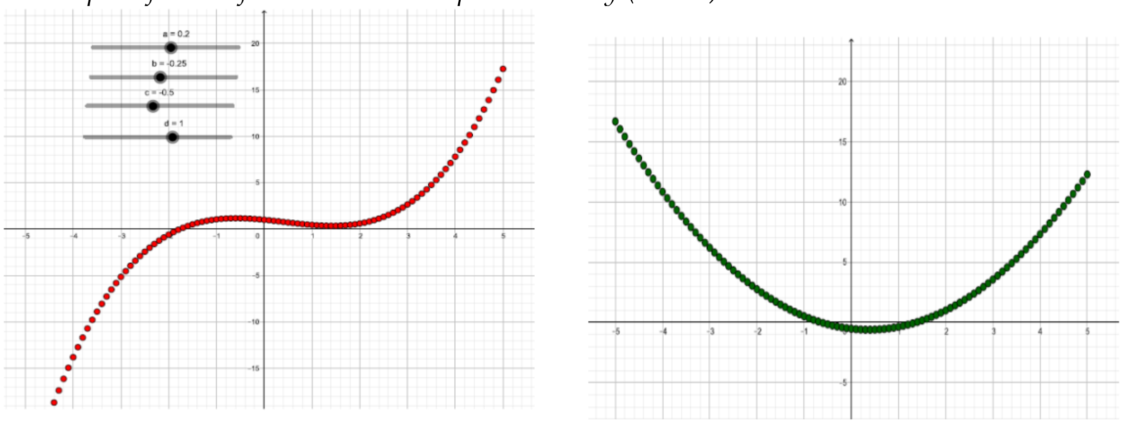


$f(z_{10}) = {z_{10}}^2 - z_{10} + 1$

The differential subtraction function of z_{10} from the f function *f*

Figure 5

The output of task 2 for the second computer activity (item 2)

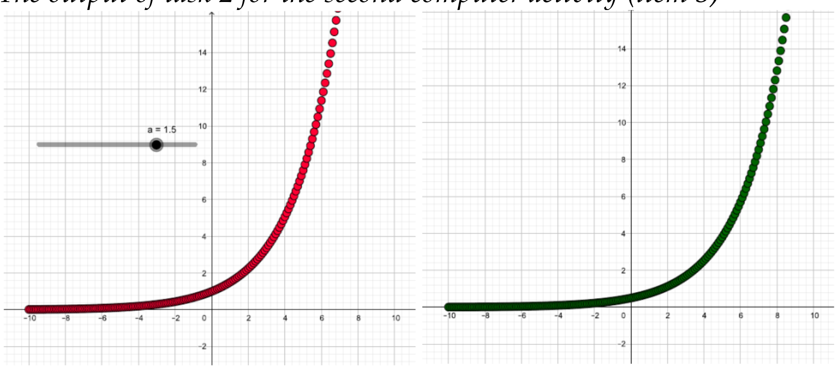


$f(z_{10}) = 0/2{z_{10}}^3 - 0/25{z_{10}}^2 - 0/5z_{10} + 1$

The differential subtraction function of z_{10} from the f function *f*

Figure 6

The output of task 2 for the second computer activity (item 3)



$E(z_{10}) = 1.5^{z_{10}}$

The differential subtraction function of z_{10} from the f

The third activity expands the second activity so that the differential fraction sequence for the real function f is obtained according to the value of x_0 for , n={1,2,3,…}. The sequence is interpreted according to the graph f of the sequence of decisive slope at a point obtained if n → ∞, the function's derivative. The details of the third activity are as follows:

Task 1: The students were instructed to write a A_3 program in Geogebra first, which receives quadratic functions, and polynomials (for example, degree 3) with domain R as input. The outputs must be the differential fraction function and the limit of the differential fraction function at infinity of each of the above functions. The mentioned method is written in Geogebra as follows:

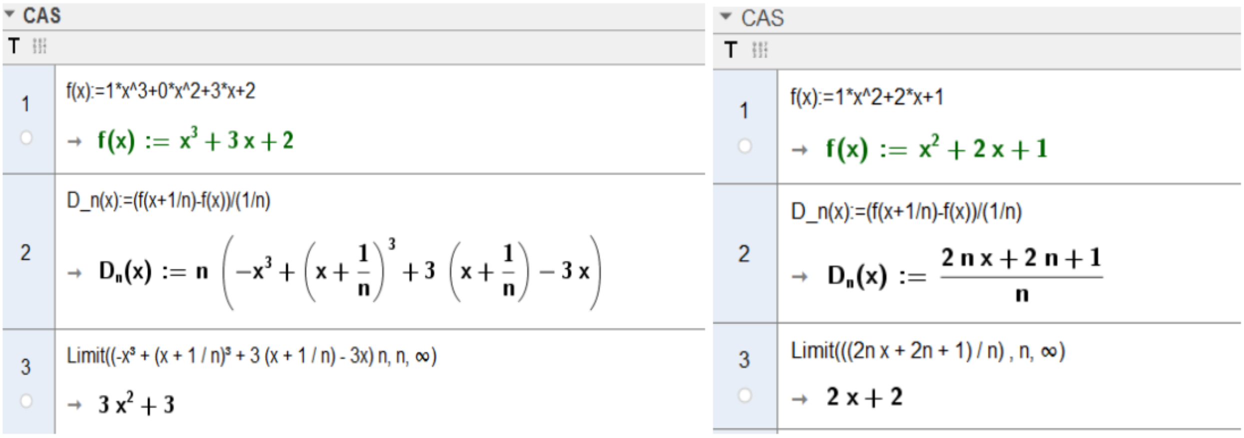
```
A3:CAS;
f(x):=a*x^2+b*x+c  and f(x):=a*x^3+b*x^2+cx+d
(f(x+1/n)-f(x))/(1/n)
Lim((f(x+1/n)-f(x))/(1/n),n,∞)
```

Task 2: Students were instructed to employ the A_3 program for the following inputs:

1) $f(x) = x^2 + 2x + 1$
2) $f(x) = x^3 + 3x + 2$

Figure 7

The output of task 2 for the third computer activity (items 1 and 2)



Cycle 2: Class Discussions. The second session was held in the classroom. Class discussions were held following the three computer-based activities, where students could express their ideas, thoughts, and understanding of each activity. In addition, they were instructed to use logic rather than geometry to solve each activity. Students could then use their comprehension of computer activities on paper. Regarding the first activity, students were asked to define $f(z)$ and differential functions. Then they were given some quadratic and cubic $f(z)$ functions and an exponential function, and they were asked to first obtain the differential function of these functions and then draw their graph and explain their differential function. For the second activity, the students were asked to define the function $f(z_{10})$ and then given some functions $f(z_{10})$ quadratic and cubic and exponential functions to calculate its differential fraction function and then draw their graphs. In the general case, they obtained the differential fraction function for ($\frac{1}{n}, n\epsilon N$) quadratic and cubic, and exponential functions and explained the differential fraction function with the Q domain. In the case of the third activity, the quadratic and cubic functions with the domain R were given, and the differential fraction function was obtained from these real functions. Then they calculated their limit at infinity and expressed their interpretation of $D_n(x_0)$.

Cycle 3: Exercises. The exercise phase in the ACE teaching cycle reinforces the previous two phases (computer activities and classroom discussion). The exercises aimed for students to expand the schema of the discrete derivative approach by working with Z-functions and their differential Z-function with the Z domain, differential fraction function with Q and R domains, and drawing their graphs. These exercises were designed as night homework for students and included different mathematics situations. Students strengthen their knowledge in computer activities and class discussions by solving exercises. Some exercises are described below.

Exercise 1 (related to the first computer activity): Draw the graph of Z functions and their differential functions $D_f(z)$ with a discrete approach.

1) $f(z) = 4z^2 - 2z + 3$
2) $f(z) = z^3 - 2z - 1$
3) $E(z) = e^z, z \in Z$

Exercise 2 (related to the second computer activity): for the Z_k function, $f(z_{10}) = 2z_{10}^2 - 5z_{10} - 3$ $(z_{10} \in Z_{10})$, first, obtain its $Df_{10}(z_{10})$ and draw the graphs of both functions with a discrete approach.

Exercise 3 (related to the third computer activity): A) if $f(x) = x^2 - 5x - 8$, calculate the derivative of the function (x) using the discrete approach. B) We have filled a cylindrical source with a vertical axis with 2000 liters of water. If we open the valve under the source, the source will be empty within 5 minutes. Suppose we have opened the source valve at the moment $t = 0$. According to one of the laws of physics, we know that the remaining water in the source after t minutes is equal to $v(t) = 2000000 - 8000t + 80t^2$. Find the instantaneous rate of change of water exit from the source in t=30 minutes.

4.4. Data Analysis

Data analysis in the present study was descriptive and inferential. In the descriptive part, tables of frequency, mean, and standard deviation were used, and in the inferential part, the covariance analysis method was utilized using SPSS software.

5. Results

Table 2 presents the descriptive statistics of group scores.

Table 2
Descriptive statistics in experimental and control groups

Group	*N*	*SD*	*Mean*
Experimental	21	1.50	17.57
Control	21	2.20	14.80

Table 2 shows the descriptive criteria related to the data in two control and experimental groups. An average of 14.80 student’s grades in teaching the discrete derivative approach in the control group was obtained traditionally, and an average of 17.57 of the student’s grades in teaching the discrete derivative method was obtained with the help of the ACE cycle. It is possible to understand the effectiveness of the ACE cycle on a better understanding of the derivative concept by comparing these two averages. However, the independent *t*-test table results further confirm this statement. First, the assumption that the variance of two groups is equal is examined by performing the independent *t*-test.

Levene’s test is considered in Table 3 to check this hypothesis. This test’s significance level (sig) equals.016, smaller than.05. The equality of variances assumption is thus disproved. The second row of Table 3 verifies that the two experimental and control groups' average scores are equal. The significance level (sig) in the *t*-test in the second row of Table 3 equals zero, which is less than .01. Therefore, a considerable exist difference between the average grades of teaching the discrete derivative approach with the help of the ACE teaching cycle in the experimental group and teaching the discrete derivative approach using the routine method in the control group. Considering that the confidence intervals obtained have the same sign, there is a significant difference between the experimental and control group results. Therefore, learning derivation using ACE cycles beyond 90% certainty allowed us to gain a deeper comprehension of the derivation concept in comparison to the control group.

Table 3

T-test of independent samples in two experimental and control groups

	Levene's test for equality of variances		*t-test for Equality of Means*						
								95% confidence interval of the difference	
	F	Sig.	*t*	df	Sig (2-tailed)	Mean difference	Mean deviation	Lower	Upper
Equality of variance	6.346	.016	−4.366	40	<.01	−2.762	0.647	−4.049	−1.476
Inequality of variance			−4.366	32.76	<.01	−2.762	0.647	−4.058	−1.466

6. Discussion and Conclusion

The current research aims at evaluating the effectiveness of the ACE teaching cycle on students' comprehension of the derivative concept with a discrete approach. As can be seen from the Table 3, a considerable difference exist between the average grades of teaching the discrete derivative approach by ACE in the Geogebra software environment in the experimental group and teaching the discrete derivative approach using the routine method in the control group. Therefore, teaching the discrete derivative approach using the ACE cycle significantly impacts students' understanding of the derivative concept. It gradually led to an improved comprehension of the change song and moment song with the help of sequences, which shows the success of the ACE teaching cycle in using Geogebra software in teaching and learning the derivative concept with a discrete approach. the findings of this research is line with that of Asiala et al. (1997), Borji et al. (2019), Borji et al. (2018), and Cottrill et al. (1996), who attributes the success of students' mental construction to the use of the ACE teaching cycle. The results show that students in the experimental group performed better qualitatively in answering questions and gained a in-depth comprehension of derived concepts. Evaluating the students' answers in the experimental group proved that that many had made a coherent schema of the derivative concept with a discrete approach. Students in the experimental group improved their conceptualization of the Z function and the differential Z function by writing computer programmes in groups using Geogebra software, which led to understanding the relationship between the function and the derivative function. The experimental group students solved the problems in groups and had the opportunity to learn better from each other and explained what they solved to each other step by step in the group. Group work allows group members to cover each other's weaknesses. Group learning allows learners to discuss with each other, ask each other questions, and support each other by helping wherever necessary. Group work allows learners to listen to their ideas, share their ideas, and question each other's thinking. Control students were not given this opportunity. In the control group, instruction time was devoted to teacher instruction and students did not take a progressive act in the instruction process because instruction was routine. About two students were very weak in basic mathematics, including the basic questions on the concepts required to learn the concept of derivatives, including function, differential function, and drawing Z functions and their differential function. Nevertheless, the ACE teaching cycle made the students' problem-solving skills in the field of derivation to be strengthened. The ACE teaching cycle with software can help teach Hesaban, especially the concept of derivative, to develop students' conceptual understanding. Therefore, the learning environment enriched with educational software can be quick feedback for students and a tool for accurate and fast visual representations of mathematical forms as a tool for students to manipulate mathematical structures and an opportunity to show the effects of these forms. The growth of students' creativity and thinking and the improvement of the learning of mathematical concepts, so the use of computer-based educational methods. Visualization and multiple representations are essential advantages of using mathematical educational software, which can strengthen the links between different mathematical concepts and create deeper learning in students. The provided feedback is a reaction to the student's learning behavior and includes verbal and non-verbal reactions, such as warning, drawing attention, and suggesting the next steps.

Therefore, math professors should be familiar with math software to develop educational tasks that facilitates students' comprehension of math concepts and motivates students to employ this technology to improve their learning. Technology increases students' learning and should be used in teaching and learning mathematics is essential (NCTM,2000).

ACE teaching cycle, along with educational software, provides a context for the innovative ideas of learners and can also play a decisive role in learners' academic progress and creativity. The results showed that this teaching cycle is more effective in developing students' thinking flexibility than the routine teaching method. This teaching cycle can enable learners to pay attention to the concept and principle of the subject and details when dealing with severe and new-conceptual-level mathematical problems. The ability of learners to expand their answers is also enhanced. Due to the effectiveness of the ACE teaching cycle, learners will experience more profound and more stable learning in this learning environment. Teachers and professors can use the ACE teaching cycle and educational software for students' deep understanding of mathematical concepts and increase divergent and creative thinking. However, more research should be conducted to help improve students' understanding of mathematical concepts through this teaching cycle.

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